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On Approximation Theory and Functional Equations*

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1. INTRODUCTION

For the past decade, I have been interested in approximation properties of functions of several variables. The study of one special case has led to some unexpectedly interesting results. Let X and Y be compact spaces and let $S = X \times Y$. As usual, $C[S]$ denotes the space of real-valued continuous functions on S with the uniform norm $\| F \| = \max | F(x, y)|$, for $(x, y) \in S$. Let H be the set of functions in $C[S]$ which can be expressed in the form $f(x, y) = A(x) + B(y)$, where $A \in C[X]$ and $B \in C[Y]$. It is easily seen that H is a closed subspace of $C[S]$. Let K be a compact subset of S and $H_{|K}$ the set of restrictions to K of functions $f \in H$. Then, $H_{|K}$ is a subspace of $C[K]$. Our central problem is the study of the relationship between $H_{|K}$ and $C[K]$.

The present paper, which is the first of a series, will show that for certain choices of K there is a close connection between the nature of $H_{|K}$ and the study of a special type of functional equations. We obtain necessary and sufficient conditions for $H_{|K}$ to coincide with or to be dense in $C[K]$. In the special case $X = Y = [0, 1]$, certain choices for K will show that $H_{|K}$ need not be closed in $C[K]$, and functions $F \in C[K]$ need not have best uniform approximations by $f \in H$. Some of the results dealing with functional equations are also new and have independent interest.

2. FUNCTIONAL EQUATIONS

Let $k(x)$, $\beta(x)$ and $u(x)$ belong to $C[0, 1]$, with β taking values in $[0, 1]$, and ask for a continuous function $\varphi \in C[0, 1]$ such that

$$\varphi(x) - k(x)\, \varphi(\beta(x)) = u(x) \tag{1}$$

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for all $x \in [0, 1]$. Many papers have dealt with this classical functional equation, mostly with severe restrictions on β and u. The easiest case to study is that where $\| k \| < 1$; for then one may apply the standard fixed point theorem for contraction mappings. The special case $k(x) \equiv 1$ is central to the general problem. (A summary of the literature is to be found in Kuczma [2, Chapters II, IV, V]. Our first results deal with generalizations of this equation.

Let α and β be continuous mappings from X into Y. Given any function $u \in C[X]$, we ask for a function $\varphi \in C[Y]$ such that

$$\varphi(\alpha(x)) - \varphi(\beta(x)) = u(x) \tag{2}$$

for all $x \in X$. Let Γ be the set of $x \in X$ for which $\alpha(x) = \beta(x)$. Clearly, (2) cannot have a solution unless u vanishes on Γ. We can then ask if any additional conditions on u are needed in order that there exists a solution for (2). We are also interested in the existence of approximate solutions of (2); given $\epsilon > 0$, is there a function $\varphi \in C[Y]$ such that

$$| \varphi(\alpha(x)) - \varphi(\beta(x)) - u(x)| < \epsilon$$

for all $x \in X$?

More generally, suppose that $\beta_0, \beta_1, \ldots, \beta_n$ are mappings from X into Y, not necessarily one-to-one, and consider the system of simultaneous equations

$$\begin{aligned} \varphi(\beta_0(x)) - \varphi(\beta_1(x)) &= u_1(x), \\ \varphi(\beta_1(x)) - \varphi(\beta_2(x)) &= u_2(x), \\ &\vdots \\ \varphi(\beta_{n-1}(x)) - \varphi(\beta_n(x)) &= u_n(x). \end{aligned} \tag{3}$$

For any i and j, $i < j$, let Γ_{ij} be the set of all $x \in X$ with $\beta_i(x) = \beta_j(x)$. It is clear that Eqs. (3) cannot admit either an exact solution φ or arbitrarily good approximate solutions φ unless the given functions u_k obey the condition

$$u_k(x) = 0 \qquad \text{for all} \quad x \in \Gamma_{k-1,k}.$$

However, these are not the only requirements that must be imposed on the functions u_i. For example, if φ obeys the first two equations in (3), then it must also satisfy

$$\varphi(\beta_0(x)) - \varphi(\beta_2(x)) = u_1(x) + u_2(x),$$

so that it is necessary that $u_1(x) + u_2(x) = 0$ on the set Γ_{02}. In general, (3) can have no solution φ unless

$$u_k(x) + u_{k+1}(x) + \cdots + u_m(x) = 0 \qquad \text{for} \quad x \in \Gamma_{k-1,m}, \tag{4}$$

for $k = 1, 2, \ldots, n$, $m = 1, 2, \ldots, n$, $k \leqslant m$.

When $X = Y = [0, 1]$, we can picture the β_i as a collection of curves in the unit square S, and the sets Γ_{ij} describe their intersection pattern. If there are multiple intersections, then the sets Γ_{ij} are not all disjoint.

In the next section, we obtain necessary and sufficient conditions for the system (3) to have a solution (or to have arbitrarily good approximate solutions) for every choice of the functions u_k obeying the set of conditions (4). In the following section, this will be applied in the case $n = 1$ (one equation and two curves) to illustrate some of the complexity of the situation.

3. The Main Theorem

Let K be the compact subset of S consisting of the graphs of the $n + 1$ curves (mappings) β_i.

Theorem 1. *The system of functional Eqs.* (3) *has a solution* $\varphi \in C[Y]$ *for every choice of the continuous functions* u_k *obeying conditions* (4) *if and only if* $H_{|K} = C[K]$; (3) *admits arbitrarily good approximate solutions if and only if* $H_{|K}$ *is dense in* $C[K]$.

Because of its interest, we also state the special case of Theorem 1 arising from $n = 1$.

Theorem 2. *Let* K *be the subset of* S *consisting of the graphs of the curves* $y = \alpha(x)$, $y = \beta(x)$, *and let* Γ *be the set of* x *with* $\alpha(x) = \beta(x)$. *Then the functional Eq.* (2) *has a continuous solution* φ *for every function* $u \in C[X]$ *with* $u_{|\Gamma} = 0$ *if and only if* $H_{|K} = C[K]$. *Equation* (2) *has approximate solutions if and only if* $H_{|K}$ *is dense in* $C[K]$.

We observe that these results reduce the solvability problem for these functional equations to the problem of determining those measures whose support lies in K and which annihilate H. We will return to this problem in future papers.

In the interest of clarity, we will first prove Theorem 2. Since this serves as a model for the proof of Theorem 1, we will not have to give all the details of the latter.

Suppose that $H_{|K}$ is dense in $C[K]$. Given any function $u \in C[X]$ with $u(x) = 0$ for all $x \in \Gamma$, define a function F on the compact set K by

$$F(x, y) = \begin{cases} u(x) & \text{if } (x, y) \text{ lies on the graph of } \alpha, \\ 0 & \text{if } (x, y) \text{ lies on the graph of } \beta. \end{cases} \tag{6}$$

Observe that the assumption that u vanishes on Γ is precisely what is needed to be sure that F is continuous on K. Given $\epsilon > 0$, choose $f \in H$, $f(x, y) = A(x) + B(y)$, so that $\|f - F\|_K < \epsilon$. Looking at this separately on the graphs of α and β, we have the inequalities

$$| A(x) + B(\alpha(x)) - u(x)| < \epsilon$$
$$| A(x) + B(\beta(x)) - 0 | < \epsilon$$

for all $x \in X$. Looking at the difference of the expressions inside the absolute value signs, we have

$$| B(\alpha(x)) - B(\beta(x)) - u(x)| < 2\epsilon$$

for all $x \in X$, and $\varphi = B$ is an approximate solution of (2).

Conversely, suppose that (2) has approximate solutions for any choice of u with $u_{|\Gamma} = 0$. Given any function $F \in C[K]$, define $u \in C[X]$ by

$$u(x) = F(x, \alpha(x)) - F(x, \beta(x)).$$

Note that $u(x) = 0$ for all $x \in \Gamma$. Given $\epsilon > 0$, let φ be a corresponding approximate solution of (2), and set

$$A(x) = F(x, \beta(x)) - \varphi(\beta(x)),$$
$$B(y) = \varphi(y).$$

Then $A \in C[X]$, $B \in C[Y]$ and $f(x, y) = A(x) + B(y) \in H$. Examining f on K, which is $\alpha \cup \beta$, we first look at α, where we find

$$\begin{aligned} f(x, y) &= f(x, \alpha(x)) = A(x) + B(\alpha(x)) \\ &= F(x, \beta(x)) - \varphi(\beta(x)) + \varphi(\alpha(x)) \\ &= -u(x) - \varphi(\beta(x)) + \varphi(\alpha(x)) + F(x, \alpha(x)) \end{aligned}$$

and thus

$$|f(x, y) - F(x, y)| = | \varphi(\alpha(x)) - \varphi(\beta(x)) - u(x)| < \epsilon.$$

Similarly, one finds that on β, $f(x, y) = F(x, y)$. Hence $\|f - F\|_K < \epsilon$, and $H_{|K}$ is dense in $C[K]$.

The same argument, with $\epsilon = 0$, shows that $H_{|K} = C[K]$ if and only if (2) has a solution φ for each $u \in C[X]$ with $u_{|\Gamma} = 0$.

We also note that this argument shows that (2) has a solution for a given u obeying the condition $u_{|\Gamma} = 0$ if and only if the corresponding function F defined by (5) lies in $H_{|K}$, and (2) has approximate solutions φ if F lies in the closure of $H_{|K}$.

We now proceed with the proof of Theorem 1. Suppose that $H_{|K}$ is dense in $C[K]$, and that we are given functions $u_1, u_2, \dots, u_n$ obeying (4). On the set $K = \bigcup_0^n \beta_i$ define a function F by

$$F(x, y) = \begin{cases} 0 & \text{if } (x, y) \text{ lies on the graph of } \beta_0, \\ -\sum_1^k u_j(x) & \text{if } (x, y) \text{ lies on the graph of } \beta_k. \end{cases} \tag{6}$$

Then, it is seen at once that the relations (4) are precisely the data needed to verify that the behavior of F on the sets Γ_{ij} is such that F is continuous on K.

By hypothesis, given any $\epsilon > 0$, there are $A \in C[X]$ and $B \in C[Y]$ such that $\|f - F\|_K < \epsilon$, where $f(x, y) = A(x) + B(y)$. On β_0 we then have

$$| A(x) + B(\beta_0(x)) - 0 | < \epsilon.$$

On β_1,

$$| A(x) + B(\beta_1(x)) + u_1(x)| < \epsilon.$$

Hence

$$| B(\beta_0(x)) - B(\beta_1(x)) - u_1(x)| < 2\epsilon.$$

Similarly, on β_k we have by (6),

$$\left| A(x) + B(\beta_k(x)) + \sum_1^k u_i(x) \right| < \epsilon,$$

and it follows that

$$| B(\beta_{k-1}(x)) - B(\beta_k(x)) - u_k(x)| < 2\epsilon.$$

Thus $\varphi = B$ is an approximate solution of the system (3).

Conversely, suppose that (3) has arbitrarily good approximate solutions for any choice of the functions u_i obeying restrictions (4). Given a function $F \in C[K]$, define n functions by

$$u_k(x) = F(x, \beta_{k-1}(x)) - F(x, \beta_k(x)).$$

Examination shows that these functions obey all the restrictions given in (4). Accordingly, given any $\epsilon > 0$, there is a function φ which solves (3) with error uniformly smaller than ϵ. Define a function $f \in H$ by $f(x, y) = A(x) + B(y)$, where

$$A(x) = F(x, \beta_0(x)) - \varphi(\beta_0(x)),$$
$$B(y) = \varphi(y).$$

Looking at f on the set $K = \bigcup_0^n \beta_j$, we find that on β_0,

$$f(x, y) = A(x) + B(\beta_0(x)) = F(x, \beta_0(x)) - \varphi(\beta_0(x)) + \varphi(\beta_0(x))$$
$$= F(x, y)$$

and on β_1,

$$f(x, y) = A(x) + B(\beta_1(x)) = F(x, \beta_0(x)) - \varphi(\beta_0(x)) + \varphi(\beta_1(x))$$
$$= F(x, \beta_0(x)) - u_1(x) - \{\varphi(\beta_0(x)) - \varphi(\beta_1(x)) - u_1(x)\}$$
$$= F(x, \beta_1(x)) - \{\varphi(\beta_0(x)) - \varphi(\beta_1(x)) - u_1(x)\}.$$

Thus, on β_1,

$$|f(x, y) - F(x, y)| \leqslant | \varphi(\beta_0(x)) - \varphi(\beta_1(x)) - u_1(x)| < \epsilon.$$

Likewise, a similar computation shows that on β_2,

$$|f(x, y) - F(x, y)| < 2\epsilon,$$

and, finally, $\|f - F\|_{\beta_n} < n\epsilon$. Thus $\|f - F\|_K < n\epsilon$ and $H_{|K}$ is dense in $C[K]$.

We note that the same proof applies if u and φ are allowed to take values in a normed linear space E.

4. Special Cases

Let $X = Y = [0, 1]$ so that S is the unit square. We shall use particular choices of curves to illustrate Theorems 1 and 2. As the first example, we choose $\alpha(x) = x/2$ and $\beta(x) = (1 + x)/2$, and apply Theorem 2. The set Γ is empty, and we are concerned with the solution of the functional equation

$$\varphi(\tfrac{1}{2}x) - \varphi(\tfrac{1}{2} + \tfrac{1}{2}x) = u(x) \tag{7}$$

for $x \in [0, 1]$ and an arbitrary function $u \in C[0, 1]$.

The general solution of (7) is easily obtained, since the equation can be used to define φ in a segmental fashion. Given any function $\theta \in C[0, 1/2]$ such that $\theta(0) - \theta(1/2) = u(0)$, set

$$\varphi(x) = \begin{cases} \theta(x) & \text{for} \quad 0 \leqslant x \leqslant 1/2, \\ \theta(x - \tfrac{1}{2}) - u(2x - 1) & \text{for} \quad 1/2 \leqslant x \leqslant 1. \end{cases}$$

Then φ satisfies (7); thus this functional equation has infinitely many solutions for any choice of u.

The set K consists of two parallel line segments given by the graphs of α

and β. According to Theorem 2, we conclude that $H_{|K} = C[K]$; moreover, we also conclude that for any $F \in C[K]$, there are infinitely many $f \in H$ such that $f = F$ on K.

In this case, it is easy to present a complete treatment of $H_{|K}$, simpler than via (7). Given any $F \in C[K]$, choose any function $A \in C[0, 1]$ such that

$$A(1) - A(0) = F(1, 1/2) - F(0, 1/2).$$

Then, set

$$B(y) = \begin{cases} F(2y, y) - A(2y) & \text{for } 0 \leqslant y \leqslant 1/2, \\ F(2y-1, y) - A(2y-1) & \text{for } 1/2 \leqslant y \leqslant 1; \end{cases}$$

the function $f(x, y) = A(x) + B(y)$ coincides with F on K. (The restriction on A assures that B is continuous at $y = 1/2$.)

As our second example, choose $\alpha(x) = x$ and $\beta(x) = x^2$. The corresponding functional equation is

$$\varphi(x) - \varphi(x^2) = u(x) \tag{8}$$

and is one that has been studied extensively (see [2, p. 66]). The set Γ consists of 0 and 1 so that u must obey $u(0) = u(1) = 0$. The following result is well known.

LEMMA 1. *A necessary condition in order that* (8) *have a continuous solution* $\varphi \in C[0, 1]$ *is that each of the series* $\sum_0^\infty u(x^{2^n})$ *and* $\sum_1^\infty u(x^{(1/2)^n})$ *shall converge for each* $x \in [0, 1]$.

This follows from (8) by repeatedly making either the substitution of x^2 for x, or $\sqrt{x}$ for x, and adding the resulting equations to obtain the pair of identities

$$\begin{aligned} \varphi(x) - \varphi(x^{2^n}) &= \sum_0^{n-1} u(x^{2^k}), \\ \varphi(x^{(1/2)^n}) - \varphi(x) &= \sum_1^{n} u(x^{(1/2)^k}). \end{aligned} \tag{9}$$

If φ is continuous at 0 and at 1, then $\lim_{n\to\infty} \varphi(x^{2^n})$ and $\lim_{n\to\infty} \varphi(x^{(1/2)^n})$ must exist for each x, and the Lemma follows.

Consider the special function $u \in C[0, 1]$ defined by

$$u(x) = \begin{cases} \dfrac{1}{\log\log(10/x)} & \text{for } 0 < x \leqslant .5, \\ \dfrac{c}{-\log\log(1/x)} & \text{for } .5 \leqslant x < 1, \end{cases}$$

with $u(0) = u(1) = 0$ and with c selected so as to force u to be continuous at $x = .5$. Examination shows that each of the series in Lemma 1 diverges for every x, $0 < x < 1$. We conclude that the functional Eq. (8) has no continuous solution φ for this choice of u and thus, in this case, $H_{|K} \neq C[K]$.

The next result seems to be new, even for the special functional equation (8).

THEOREM 3. *Let β be a continuous increasing function obeying $\beta(0) = \beta(1) = 0$ and $0 < \beta(x) < x$ for $0 < x < 1$. Then, the functional equation*

$$\varphi(x) - \varphi(\beta(x)) = u(x) \tag{10}$$

has arbitrarily good approximate solutions for every choice of $u \in C[0, 1]$ obeying $u(0) = u(1) = 0$.

Given u, let u_N be selected so that $\| u_N \| = \| u \|$, $u_N(x) = 0$ for $0 \leqslant x \leqslant 1/N$ and $N/(N+1) \leqslant x \leqslant 1$, and $\| u_N - u \| \to 0$ as $N \to \infty$. Choose a positive number $\rho < 1$ and, motivated by (9), set

$$\varphi(x) = \sum_0^\infty \rho^n u_N(\beta^n(x)),$$

where $\beta^0(x) \equiv x$, $\beta^1(x) \equiv \beta(x)$ and, for $n = 2, 3,\ldots, \beta^n(x) = \beta(\beta^{n-1}(x))$. Note that $\varphi \in C[0, 1]$ and that, on $[0, 1]$,

$$\varphi(x) - \varphi(\beta(x))\rho = u_N(x).$$

LEMMA 2. *For any N, there is a number C depending on β but not on ρ, such that $\| \varphi \| < (1 - \rho^C) \| u \| (1 - \rho)^{-1}$.*

Because u_N vanishes on a neighborhood of 0 and of 1, $u_N(\beta^n(x)) = 0$ when $\beta^n(x) < 1/N$ and when $\beta^n(x) > N/(N+1)$. Hence, for each x, the series (11) is really a finite sum, with n running from some index n_1 to an index n_2. These can be taken as characterized by $\beta^{n_1}(x) \sim N/(N+1)$ and $\beta^{n_2}(x) \sim 1/N$. Thus, for any x, we have $\beta^{n_2 - n_1}(N/(N+1)) \sim 1/N$. Hence, there is an integer C, determined by $\beta^C(N/(N+1)) < 1/N$, such that the series (11) has never more than C nonzero terms, no matter which $x \in [0, 1]$ is selected. For example, when $\beta(x) = x^m$, C can be taken to be $1 + [\log(n \log N)/\log m]$. Consequently, for any x,

$$\begin{aligned} | \varphi(x)| &\leqslant (\rho^{n_1} + \cdots + \rho^{n_1 + C - 1}) \| u_N \| \\ &\leqslant \frac{(1 - \rho^C)}{1 - \rho} \| u \|. \end{aligned}$$

To complete the proof of Theorem 3, suppose $\epsilon > 0$ is given. Choose N

so that $\| u_N - u \| < \epsilon$, and then choose ρ close to 1 so that $1 - \rho^C < \epsilon/\| u \|$. Observe that, for any x,

$$\varphi(x) - \varphi(\beta(x)) - u(x) = u_N(x) - u(x) + (\rho - 1)\, \varphi(\beta(x))$$

so that

$$\begin{aligned} | \varphi(x) - \varphi(\beta(x)) - u(x)| &< \| u_N - u \| + (1 - \rho) \| \varphi \| \\ &< \| u_N - u \| + (1 - \rho^C) \| u \| \\ &< 2\epsilon. \end{aligned}$$

Hence, φ is an approximate solution of (10). Note that the proof holds also for any increasing function β having a finite set Γ of fixed points.

COROLLARY 1. *Let K be the subset of the unit square consisting of the curves $y = x$ and $y = x^2$. Then $H_{|K}$ is a proper dense subspace of $C[K]$.*

We note that, in this case, there are functions $F \in C[K]$ that do not have best uniform approximations by functions in H. This seems to be contrary to one of the assertions in Cereteli [1].

A second deduction from Theorem 3 may also be of independent interest.

COROLLARY 2. *Let $\mu_n = \int_0^1 t^n\, d\psi(t)$, $n = 0, 1, \ldots$, where ψ is of bounded variation. If $\mu_n = \mu_{pn}$ for $n = 1, 2, \ldots$ and some integer p, then $\mu_1 = \mu_2 = \cdots$.*

This results from the fact that the linear functional L defined by $d\psi$ will annihilate all polynomials of the form $Q(t) = P(t) - P(t^p)$, and hence all continuous functions for which $f(0) = f(1) = 0$. But such a functional must be a linear combination of point masses at 0 and at 1.

Finally, we give a simple example to show that $H_{|K}$ need not be dense in $C[K]$. With $\alpha(x) = x$, choose $\beta(x) = 9x^3 - (27/2)\, x^2 + (11/2)x$. The set Γ is $\{0, 1/2, 1\}$. Thus, $H_{|K}$ would be dense in $C[K]$ if the functional equation

$$\varphi(x) - \varphi(\beta(x)) = u(x) \tag{12}$$

has approximate solutions for every continuous function u obeying $u(0) = u(1/2) = u(1) = 0$. However, it is easily seen that no solution can exist if $u(x)$ does not also obey the relation $u(1/3) + u(2/3) = 0$. For, putting $x = 1/3$ and then $x = 2/3$ in (12), and using the fact that $\beta(1/3) = 2/3$ and $\beta(2/3) = 1/2$, we have

$$\begin{aligned} \varphi(1/3) - \varphi(2/3) &\sim u(1/3), \\ \varphi(2/3) - \varphi(1/3) &\sim u(2/3), \end{aligned}$$

which must hold with arbitrary accuracy. Thus, in this case, the closure of $H_{|K}$ is a proper subspace of $C[K]$. (This can also be seen by exhibiting a measure on K which annihilates H but not $C[K]$.)

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